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RODICA D. COSTIN, MARINA DAVID

**ABSTRACT.** Representation of analytic functions as convergent series in Jacobi polynomials  $P_n^{(\alpha, \beta)}$  is reformulated using a unified approach for all  $\alpha, \beta \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ . The coefficients of the series are given as usual integrals in the classical case (when  $\Re \alpha, \Re \beta > -1$ ), or by the Hadamard principal part of these integrals when they diverge.

As an application it is shown that inhomogeneous hypergeometric equations do generically have a unique solution which is analytic at both singular points in  $\mathbb{C}$ .

## 1. INTRODUCTION

The purpose of the present note is to provide a unified and easy to apply result on expansion of analytic functions in Jacobi series. Such expansions are useful in approximation theory, as well as in a variety of other areas, such as the study of differential equations (see e.g. [4], [5]), non-selfadjoint spectral problems which appear in the study of stability of nonlinear partial differential equations. An application to one such problem is given in Section §3.

**1.1. An overview of Jacobi polynomials for classical parameters.** We summarize below a few well known facts about Jacobi polynomials  $P_n^{(\alpha, \beta)}(z)$ ,  $n = 0, 1, 2, \dots$  for  $\alpha, \beta > -1$ . (See [9] or [13] for an overview).

The Jacobi polynomials are determined by the condition of mutual orthogonality on the interval  $[-1, 1]$  with respect to the weight

$$W(z) = (1 - z)^\alpha (1 + z)^\beta$$

namely

$$(1) \quad \int_{-1}^1 P_n^{(\alpha, \beta)}(z) P_k^{(\alpha, \beta)}(z) W(z) dz = \mathcal{A}_n \delta_{n,k}$$

where

$$\mathcal{A}_0 = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$$

and

$$(2) \quad \mathcal{A}_n = 2^{\alpha+\beta+1} \frac{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1) n!} \quad \text{for } n > 0$$

Relation (1) and the condition that the coefficient of the leading term,  $z^n$  in  $P_n^{(\alpha,\beta)}(z)$  be positive uniquely determine the Jacobi polynomials.

In the particular case when  $\alpha = \beta$  these polynomials are called Gegenbauer (or *ultraspherical*), and when  $\alpha, \beta = \pm \frac{1}{2}$  they are called Chebyshev polynomials.

As any orthogonal polynomials, Jacobi polynomials satisfy a three term recurrence relation

$$(3) \quad P_{n+1}^{(\alpha,\beta)}(z) = (A_n z + B_n) P_n^{(\alpha,\beta)}(z) - C_n P_{n-1}^{(\alpha,\beta)}(z)$$

where  $P_0^{(\alpha,\beta)}(z) = 1$ ,  $P_1^{(\alpha,\beta)}(z) = A_0 z + B_0$  and

$$\begin{aligned} A_n &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)} \\ B_n &= \frac{(\alpha^2 - \beta^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \\ C_n &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \end{aligned}$$

Also, the Jacobi polynomials satisfy the Rodrigues formula:

$$(4) \quad P_n^{(\alpha,\beta)}(z) = D_n \frac{1}{W(z)} \frac{d^n}{dz^n} [Q(z)^n W(z)]$$

where

$$(5) \quad W(z) = (1-z)^\alpha (1+z)^\beta, \quad Q(z) = (1-z)(1+z), \quad D_n = \frac{(-1)^n}{2^n n!}$$

Furthermore,  $P_n^{(\alpha,\beta)}(z)$  satisfy the differential equation:

$$(6) \quad (1-z^2)P'' + [\beta - \alpha - (\alpha + \beta + 2)z]P' + n(n + \alpha + \beta + 1)P = 0$$

i.e.  $P_n^{(\alpha,\beta)}(z)$  are eigenfunctions of a self-adjoint operator.

It is a classic result that analytic functions can be expanded in Jacobi series (see [12] p.245):

**Theorem 1.** *Let  $\alpha > -1$ ,  $\beta > -1$ . Let  $f$  be analytic in the interior of an ellipse with foci at  $\pm 1$ . Denote by  $\mathcal{E}$  the greatest such ellipse. Then  $f$  has an expansion*

$$(7) \quad f(z) = \sum_{n=0}^{\infty} f_n P_n^{(\alpha,\beta)}(z)$$

*which is convergent in the interior of  $\mathcal{E}$  and divergent outside.*

Of course, due to the orthogonality relation (1) the coefficients  $f_n$  in (7) can be calculated as

$$(8) \quad f_n = \mathcal{A}_n^{-1} \int_{-1}^1 f(z) P_n^{(\alpha, \beta)}(z) W(z) dz$$

**1.2. Jacobi polynomials for general parameters.** For  $\alpha, \beta \in \mathbb{C}$  with  $\Re \alpha, \Re \beta > -1$  all the results presented above still hold - except that now  $P_n^{(\alpha, \beta)}(z)$  are eigenfunctions of a non-selfadjoint operator.

For other complex values of  $\alpha, \beta$  however, the integrals in (1) and in (8) diverge and the Jacobi polynomials can no longer be thought of being an orthogonal system. Nevertheless, Jacobi polynomials can be defined through the three-terms recurrence, or through the Rodrigues formula.

Many decades after Theorem 1 has been established, in his 1974 paper [2], Carlson showed that expansions (7) hold for general complex parameters  $\alpha, \beta$  provided that  $\alpha + \beta \neq -2, -3, -4, \dots$ . The formulas in [2] use an integral kernel and are not easy to use in practice (see Theorem 3 in §2.3).

In 1997, in [6] it was shown that for  $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_-$  with  $\alpha + \beta \neq -2, -3, -4, \dots$  the Jacobi polynomials do satisfy (1) only now the integral should be understood as its Hadamard finite part: (1) taken in the Hadamard finite part is the bilinear form with respect to which the Jacobi polynomials are orthogonal (whose existence is guaranteed by Favard's Theorem). It was also shown in [6] that the Hadamard finite part of the integral is the analytic continuation in  $\alpha, \beta$  of the bilinear form (1) for the cases when the integral no longer converges.

**1.3. New results.** We reformulate both the classical Theorem 1 and Carlson's result from a unified point of view, easy to use in applications: Theorem 2 states that even in the case of general complex parameters (1) and (8) are valid provided that possible divergent integrals are replaced by their Hadamard finite part. A brief introduction on the Hadamard finite part is found in the Appendix, §4.

The result is formulated for Jacobi polynomials on the interval  $[0, 1]$ :

**Theorem 2.** *Let  $\alpha, \beta \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ .*

*Consider the (non-normalized) Jacobi polynomials on  $[0, 1]$  given by the Rodrigues formula:*

$$(9) \quad p_n = \frac{1}{w} \frac{d^n}{dx^n} (q^n w) \quad \text{where } w(x) = x^\alpha (1-x)^\beta, \quad q(x) = x(1-x)$$

(i) Orthogonality: *The polynomials  $p_n$  are orthogonal with respect to a bilinear form:*

$$(10) \quad \oint_0^1 p_n(x) p_k(x) w(x) dx = \delta_{nk} a_n$$

where  $\oint$  denotes the Hadamard finite part of the integral and

$$(11) \quad a_n = \frac{(n!)^2}{2^{\alpha+\beta+1}} \mathcal{A}_n = \frac{n! \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) \Gamma(n + \alpha + \beta + 1)}$$

(ii) Completeness: *Let  $\Omega \in \mathbb{C}$  be an open elliptic disk with foci 1 and 0, and let  $f$  be analytic on  $\Omega$ . Then  $f(x)$  has the expansion*

$$(12) \quad f(x) = \sum_{n=0}^{\infty} f_n p_n(x), \quad x \in \Omega$$

where

$$(13) \quad f_n = \left( \oint_0^1 p_n^2 w \right)^{-1} \oint_0^1 f p_n w$$

The expansion (12) converges absolutely and uniformly on every compact set in  $\Omega$ .

The proof of Theorem 2 is contained in §2. Section §3 uses this result to investigate existence of eigenfunctions for some non-self-adjoint operators; more precisely, we establish existence of unique analytic solutions of inhomogeneous hypergeometric equations under precise conditions on the parameters.

## 2. PROOF OF THEOREM 2

### 2.1. Jacobi polynomials on $[0, 1]$ . Substituting

$$(14) \quad p_n(x) = D_n^{-1} (-2)^{-n} P_n^{(\alpha, \beta)}(1 - 2x)$$

in the Rodriguez formula (4) we obtain (9) which shows that the polynomials defined by (9) are indeed Jacobi polynomials (up to multiplicative constants).

It is easy to see that the leading coefficient of  $p_n$  is the coefficient of  $x^n$  which equals

$$k_n = (-1)^n (n + \alpha + \beta + 1)_n = (-1)^n (n + \alpha + \beta + 1)(n + \alpha + \beta + 2) \dots (2n + \alpha + \beta)$$

(Note that if  $\alpha + \beta$  is a negative integer these polynomials are no longer a complete system).

It is also easy to see that  $p_n$  depend polynomially on  $\alpha$  and  $\beta$ .

**2.2. Orthogonality for general parameters.** Formula (10) was established in [6]. Alternatively, one can reason as follows. For  $\Re\alpha, \Re\beta > -1$  substituting (14) in (1) implies

$$(15) \quad \int_0^1 p_n(x) p_k(x) w(x) dx = 0 \quad \text{if } n \neq k$$

relation which, by analytic continuation, then holds for all other complex values of  $\alpha, \beta$ , implying (10).

Formula (11) can be obtained by a straightforward calculation based on (14) and (1), (2).

**2.3. Completeness.** Since it was established in [6] that analytic continuation in  $\alpha, \beta$  of integrals of the type (1), (8) is given by their Hadamard finite part, then Theorem 2 (ii) follows by analytic continuation of Theorem 1.

We will however give a careful proof based on the result in [2], which is the following:

**Theorem 3.** (*Theorem 1.1 in [2] for  $r = 1, s = 0$* ) *Let  $\Omega$  be an open elliptic disk with foci 1 and 0, and let  $f$  be analytic on  $\Omega$ . Let  $(\alpha, \beta) \in \mathbb{C}^2$  and assume  $\alpha + \beta \neq -2, -3, -4, \dots$ . Then, for every  $z \in \Omega$ ,*

$$(16) \quad f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} c_n R_n(z)$$

where  $R_n(z)$  are scalar multiples of  $P_n^{(\alpha, \beta)}(\frac{1-z}{2})$ , given by the formula  $R_n(z) = R_n(-\alpha - n, -\beta - n, z - 1, z)$  where

$$(17) \quad R_n(b, b', x, y) = \frac{1}{B(b, b')} \int_0^1 [ux + (1-u)y]^n u^{b-1} (1-u)^{b'-1} du$$

for  $\Re b, \Re b' > 0$  and its analytic continuation for other complex values of  $b, b'$ , and  $c_n = F^{(n)}(1 + \alpha + n, 1 + \beta + n; 1, 0)$  where

$$(18) \quad F^{(n)}(b, b'; 1, 0) = n!(2\pi i)^{-1} \int_{\gamma} f(w) R_{-n-1}(b, b'; w - 1, w) dw$$

where  $\gamma$  is a rectifiable Jordan curve in  $\Omega$  which incircles the segment  $[0, 1]$  in the positive direction.

The series (16) converges absolutely on  $\Omega$ , uniformly on every compact set in  $\Omega$ .

A straightforward calculation can be also made to link (12) and (16), yielding

$$p_n = (-1)^n (\alpha + \beta + 2n)_n R_n(x)$$

and

$$\frac{1}{n!} c_n (-1)^n (\alpha + \beta + 2n)_n^{-1} = \left( \int_0^1 p_n^2 w \right)^{-1} \int_0^1 f p_n w := f_n$$

where  $R_n(x)$  are given by (17),  $(a)_n$  denotes the rising factorial:  $(a)_n = a(a+1)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ . We illustrate below the main steps of the calculation.

In [2]  $R_n$  is shown to be a multiple of a Jacobi polynomial by using the binomial formula in (17), then integrating term by term (this is also correct if one uses the Hadamard principal part in case the integrals diverge). Then one compares with the expression of  $p_n$  obtained by using Leibniz's rule to expand its Rodrigues formula.

Integration by parts also holds for the Hadamard finite part, and  $p_n$  were proved to be an orthogonal set in [6]. It only remains to calculate the "norm" of  $p_n$  in this sense.

We first calculate the dominant term in  $p_n$ : it is found by retaining only the highest powers of  $x$  in its Rodrigues formula, yielding:  $p_n = D_n x^n + \dots$  where  $D_n = (-1)^n (\alpha + \beta + 2n)_n$ .

Then, using the fact that  $p_n$  is orthogonal to all the polynomials of degree less than  $n$ , and then using the Rodrigues formula and integrating by parts  $n$  times we obtain

$$\begin{aligned} (19) \quad \int_0^1 p_n^2 w &= D_n \int_0^1 p_n x^n w = D_n \int_0^1 [Q^n w]^{(n)} x^n \\ &= D_n (-1)^n n! \int_0^1 x^{n+\alpha} (1-x)^{n+\beta} dx = D_n (-1)^n n! B(n+\alpha+1, n+\beta+1) \end{aligned}$$

□

We will need the differential equation (6) transcribed for the Jacobi polynomials on  $[0, 1]$ :

**Proposition 4.** *The Jacobi polynomials  $p_n$  defined by (9) for general  $\alpha, \beta$  satisfy*

$$(20) \quad p'' + \left( \frac{\alpha+1}{x} + \frac{\beta+1}{x-1} \right) p' + \frac{n(n+\alpha+\beta+1)}{x(1-x)} p = 0$$

The proof is a straightforward calculation using (14) in (6). □

### 3. ANALYTIC SOLUTIONS OF INHOMOGENEOUS HYPERGEOMETRIC EQUATIONS

The hypergeometric equation

$$(21) \quad w''(x) + \left( \frac{a}{x} + \frac{b}{x-1} \right) w'(x) + \frac{c}{x(1-x)} w(x) = 0$$

does have solutions analytic at the singular point  $x = 0$  and solutions analytic at  $x = 1$  (as it is well known, and easy to establish using Frobenius theory). But, for generic parameters  $a, b, c$  no non-zero solution is analytic at both singular points  $x = 0$  and  $x = 1$ .

We investigate if an inhomogeneous version, eq. (22) below, has solutions that are analytic at both  $x = 0$  and  $x = 1$ . This type of questions appear as generalized eigenfunction problems in many types of applications, for instance in the study of blow-up of wave maps (see e.g. [1], [3]).

We find the answer to be positive:

**Theorem 5.** *Consider the following hypergeometric equation with an inhomogeneous term:*

$$(22) \quad u''(x) + \left( \frac{a}{x} + \frac{b}{x-1} \right) u'(x) + \frac{c}{x(1-x)} u(x) = \frac{g(x)}{x(1-x)}$$

where  $g$  is a function that is analytic in an elliptic domain  $\mathcal{E}$  with foci at  $0, 1$ .

Assume that:  $a, b \in \mathbb{C} \setminus \{1, 0, -1, -2, -3, \dots\}$  and

$$(23) \quad c \neq n(n+a+b-1) \quad \text{for all } n = 0, 1, 2, 3, \dots$$

Then equation (22) has a unique solution which is analytic at both  $x = 0$  and  $x = 1$ , and therefore it is analytic on the whole domain where  $g$  is analytic.

*Proof.*

Let  $p_n$  be the Jacobi polynomials on  $[0, 1]$  defined by (9) with  $\alpha = a - 1$ ,  $\beta = b - 1$ .

Denote

$$(24) \quad \lambda_n = n(n+a+b-1)$$

Then by (20) each polynomial  $p_n$  satisfies

$$(25) \quad p_n''(x) + \left( \frac{a}{x} + \frac{b}{x-1} \right) p_n'(x) = -\frac{\lambda_n}{x(1-x)} p_n(x)$$

which implies that

$$(26) \quad p_n''(x) + \left( \frac{a}{x} + \frac{b}{x-1} \right) p_n'(x) + \frac{c}{x(1-x)} p_n(x) = \frac{c - \lambda_n}{x(1-x)} p_n(x)$$

so that  $p_n(x)$  is a solution of (22) for the case when  $g(x) = (c - \lambda_n)p_n(x)$ .

We are thus led to the idea of expanding the inhomogeneous term  $g$  as a series in Jacobi polynomials.

By Theorem 2 there exist constants  $g_0, g_1, g_2, \dots$  so that  $g(x) = \sum_{n=0}^{\infty} g_n p_n(x)$  with the series converging uniformly on every compact contained in  $\mathcal{E}$  and  $g_n$  given by the formula

$$(27) \quad g_n = \left( \oint_0^1 p_n^2 w \right)^{-1} \oint_0^1 g p_n w$$

We look for a solution  $u(x)$  of (22) which is analytic at both 0 and 1, and therefore it is analytic on  $\mathcal{E}$ ; also by Theorem 2, such a solution has an expansion  $u(x) = \sum_{n=0}^{\infty} u_n p_n(x)$  which plugged in the equation (22) gives

$$(28) \quad \sum_{n=0}^{\infty} u_n \left[ p_n'' + \left( \frac{a}{x} + \frac{b}{x-1} \right) p_n' + \frac{c}{x(1-x)} p_n \right] = \sum_{n=0}^{\infty} \frac{g_n}{x(1-x)} p_n(x)$$

which by (26) implies

$$\sum_{n=0}^{\infty} \frac{c - \lambda_n}{x(1-x)} u_n (c - \lambda_n) p_n(x) = \sum_{n=0}^{\infty} \frac{g_n}{x(1-x)} p_n(x)$$

and using (10) it follows that  $u_n (c - \lambda_n) = g_n$  for all  $n$ . In view of (23) and (24) it follows that

$$(29) \quad u_n = \frac{g_n}{c - n(n + a + b - 1)}, \quad n = 0, 1, 2, \dots$$

We thus found a unique solution analytic on  $\mathcal{E}$  of (22) as the sum of the Jacobi series

$$(30) \quad u(x) = \sum_{n=0}^{\infty} \frac{g_n}{c - n(n + a + b - 1)} p_n(x)$$

with  $g_n$  given by (27). The series is obviously convergent everywhere the series of  $g$  is.  $\square$



## 4. APPENDIX: THE HADAMARD FINITE PART

The concept of the *finite part* of a (possibly divergent) integral was introduced by Hadamard [8] as a convenient way to express solutions of differential equations. He showed that this finite part of an integral (which coincides with the usual value if the integral is convergent) can be combined and manipulated in much the same way as usual integrals: they are additive on the interval of integration, changes of variable are allowed, etc. (They do not behave well with respect to inequalities.) The finite part can be calculated either by Taylor series, or by integration along closed paths in the complex plane.

Subsequently the Hadamard finite part has been interpreted in terms of distributions (see, e.g. [11]) and it turned out that many problems of mathematical physics have solutions expressible as the Hadamard finite part of (divergent) integrals, and numerical methods of calculations have been subsequently developed (see for example [7]).

The present section contains some properties of the Hadamard finite part of integrals of the type  $\int_0^x t^{\alpha-1} f(t) dt$  with  $f$  analytic at 0; when  $\Re \alpha \leq 0$ ,  $\alpha \notin (-\mathbb{N})$ , its Hadamard finite part is denoted here by

$$\oint_0^x t^{\alpha-1} f(t) dt$$

Consider  $f(x)$  a function analytic at  $x = 0$ :  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  a series with nonzero radius of convergence  $R$ . The integral

$$(31) \quad x^{-\alpha} \int_0^x t^{\alpha-1} f(t) dt$$

converges for  $\Re \alpha > 0$  and for  $|x| < R$  we have

$$(32) \quad \begin{aligned} x^{-\alpha} \int_0^x t^{\alpha-1} f(t) dt &= x^{-\alpha} \int_0^x t^{\alpha-1} \sum_{n=0}^{\infty} c_n t^n dt = x^{-\alpha} \sum_{n=0}^{\infty} c_n \int_0^x t^{n+\alpha-1} dt \\ &= x^{-\alpha} \sum_{n=0}^{\infty} \frac{c_n}{n+\alpha} x^{n+\alpha} = \sum_{n=0}^{\infty} \frac{c_n}{n+\alpha} x^n \end{aligned}$$

and we see that the final result is analytic at  $x = 0$ . For all other complex values of  $\alpha \neq 0, -1, -2, \dots$ , the Hadamard finite part of the integral is defined by:

$$x^{-\alpha} \oint_0^x t^{\alpha-1} f(t) dt := \sum_{n=0}^{\infty} \frac{c_n}{n+\alpha} x^n \quad \text{for all } \alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

This definition represents the analytic continuation in  $\alpha$  of the function (31) to  $\alpha$  in the complex plane (when  $\alpha$  is a negative integer the continuation has poles of order one).

*Remark.* For practical calculations one does not need to expand  $f$  in its full power series, just a finite number of terms are needed. For example, to calculate

$$x^{1/2} \int_0^x t^{-3/2} f(t) dt$$

first write  $f(t) = f(0) + t\tilde{f}(t)$  where  $\tilde{f}$  is analytic at 0. Then

$$x^{1/2} \int_0^x t^{-3/2} f(t) dt = x^{1/2} \int_0^x t^{-3/2} f(0) dt + x^{1/2} \int_0^x t^{-1/2} \tilde{f}(t) dt$$

where the first term is

$$x^{1/2} \int_0^x t^{-3/2} f(0) dt = x^{1/2} \frac{x^{-1/2}}{-1/2} f(0) = -2f(0)$$

and the last term is an usual integral.

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